

# A convergent finite element approximation for the quasi-static Maxwell–Landau–Lifshitz–Gilbert equations \*

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## Abstract

We propose a  $\theta$ -linear scheme for the numerical solution of the quasi-static Maxwell–Landau–Lifshitz–Gilbert (MLLG) equations. Despite the strong non-linearity of the Landau–Lifshitz–Gilbert equation, the proposed method results in a linear system at each time step. We prove that as the time and space steps tend to zero (with no further conditions when  $\theta \in (\frac{1}{2}, 1]$ ), the finite element solutions converge weakly to a weak solution of the MLLG equations. Numerical results are presented to show the applicability of the method.

**Key words:** Maxwell–Landau–Lifshitz–Gilbert, finite element, ferromagnetism

**AMS subject classifications:** 65M12, 35K55

## 1 Introduction

The Maxwell–Landau–Lifshitz–Gilbert (MLLG) equations describe the electromagnetic behavior of a ferromagnetic material. In this paper, for simplicity, we suppose that there is a bounded cavity  $\tilde{D} \subset \mathbb{R}^3$  (with perfectly conducting outer surface  $\partial\tilde{D}$ ) into which a ferromagnet  $D$  is embedded. We assume further that  $\tilde{D} \setminus \bar{D}$  is a vacuum. We will consider the quasi-static case of the MLLG system. Letting  $D_T := (0, T) \times D$  and  $\tilde{D}_T := (0, T) \times \tilde{D}$ , the magnetization field  $\mathbf{m} : D_T \rightarrow \mathbb{S}^2$ , where  $\mathbb{S}^2$  is the unit sphere in  $\mathbb{R}^3$ , and the magnetic field  $\mathbf{H} : \tilde{D}_T \rightarrow \mathbb{R}^3$  satisfy

$$\mathbf{m}_t = \lambda_1 \mathbf{m} \times \mathbf{H}_{\text{eff}} - \lambda_2 \mathbf{m} \times (\mathbf{m} \times \mathbf{H}_{\text{eff}}) \quad \text{in } D_T, \quad (1.1)$$

$$\mu_0 \mathbf{H}_t + \sigma \nabla \times (\nabla \times \mathbf{H}) = -\mu_0 \tilde{\mathbf{m}}_t \quad \text{in } \tilde{D}_T, \quad (1.2)$$

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in which  $\lambda_1 \neq 0$ ,  $\lambda_2 > 0$ ,  $\sigma \geq 0$ , and  $\mu_0 > 0$  are constants. Here  $\widetilde{\mathbf{m}} : \widetilde{D}_T \rightarrow \mathbb{R}^3$  is the zero extension of  $\mathbf{m}$  onto  $\widetilde{D}_T$ , i.e.,

$$\widetilde{\mathbf{m}}(t, \mathbf{x}) = \begin{cases} \mathbf{m}(t, \mathbf{x}), & (t, \mathbf{x}) \in D_T \\ 0, & (t, \mathbf{x}) \in \widetilde{D}_T \setminus D_T. \end{cases}$$

For simplicity the effective field  $\mathbf{H}_{\text{eff}}$  is taken to be  $\mathbf{H}_{\text{eff}} = \Delta \mathbf{m} + \mathbf{H}$ .

The system (1.1)–(1.2) is supplemented with initial conditions

$$\mathbf{m}(0, \cdot) = \mathbf{m}_0 \text{ in } D \quad \text{and} \quad \mathbf{H}(0, \cdot) = \mathbf{H}_0 \text{ in } \widetilde{D}, \quad (1.3)$$

and boundary conditions

$$\partial_n \mathbf{m} = 0 \text{ on } (0, T) \times \partial D \quad \text{and} \quad (\nabla \times \mathbf{H}) \times \mathbf{n} = 0 \text{ on } (0, T) \times \partial \widetilde{D}. \quad (1.4)$$

The equation (1.1) is the first dynamical model for the precessional motion of a magnetization, suggested by Landau and Lifshitz [12]. The existence and uniqueness of a *local* strong solution of (1.1)–(1.4) is shown by Cimrák [8]. He also proposes [7] a finite element method to approximate this local solution and provides error estimation.

Gilbert introduces a different approach for description of damped precession in [9]:

$$\lambda_1 \mathbf{m}_t + \lambda_2 \mathbf{m} \times \mathbf{m}_t = \mu \mathbf{m} \times \mathbf{H}_{\text{eff}}, \quad (1.5)$$

in which  $\mu = \lambda_1^2 + \lambda_2^2$ . A proof of the equivalence between (1.5) and (1.1) can be found in [13]. It is easier to numerically solve (1.5) than (1.1) because the latter has a double cross term, namely  $\mathbf{m} \times (\mathbf{m} \times \mathbf{H}_{\text{eff}})$ .

Instead of solving (1.1)–(1.4), Bañas, Bartels and Prohl [2] propose an implicit nonlinear scheme to solve problem (1.2)–(1.5), and prove that the finite element solution converges to a weak *global* solution of the problem. Their method requires a condition on the time step  $k$  and space step  $h$  (namely  $k = O(h^2)$ ) for the convergence of the *nonlinear* system of equations resulting from the discretization.

Following the idea developed by Alouges and Jaisson [1] for the Landau–Lifshitz–Gilbert (LLG) equation (1.5), we propose a  $\theta$ -linear finite element scheme to find a weak global solution to (1.2)–(1.5). We prove that the numerical solutions converge to a weak solution of the problem with no condition imposed on time step and space step as  $\theta \in (\frac{1}{2}, 1]$ . It is required that  $k = o(h^2)$  when  $\theta \in [0, \frac{1}{2})$ , and  $k = o(h)$  when  $\theta = \frac{1}{2}$ . The implementation aspect of the algorithm is reported in [13] where no convergence analysis is carried out.

The paper is organized as follows. Weak solutions of the MLLG equations are defined in Section 2. We also introduce in this section the  $\theta$ -linear finite element scheme. Some technical lemmas are presented in Section 3. In Section 4, we prove that the finite element solutions converge to a weak solution of the MLLG equations. Numerical experiments are presented in the last section.

## 2 Weak solutions and finite element schemes

Before presenting the definition of a weak solution to the MLLG equations, it is necessary to introduce some function spaces and to assume some conditions on the initial functions  $\mathbf{m}_0$  and  $\mathbf{H}_0$ .

The function spaces  $\mathbb{H}^1(D, \mathbb{R}^3)$  and  $\mathbb{H}(\text{curl}; \tilde{D})$  are defined as follows:

$$\begin{aligned}\mathbb{H}^1(D, \mathbb{R}^3) &= \left\{ \mathbf{u} \in \mathbb{L}^2(D, \mathbb{R}^3) : \frac{\partial \mathbf{u}}{\partial x_i} \in \mathbb{L}^2(D, \mathbb{R}^3) \text{ for } i = 1, 2, 3. \right\}, \\ \mathbb{H}(\text{curl}; \tilde{D}) &= \left\{ \mathbf{u} \in \mathbb{L}^2(\tilde{D}, \mathbb{R}^3) : \nabla \times \mathbf{u} \in \mathbb{L}^2(\tilde{D}, \mathbb{R}^3) \right\}.\end{aligned}$$

Here, for a domain  $\Omega \subset \mathbb{R}^3$ ,  $\mathbb{L}^2(\Omega, \mathbb{R}^3)$  is the usual space of Lebesgue squared integrable functions defined on  $\Omega$  and taking values in  $\mathbb{R}^3$ . Throughout this paper, we denote

$$\langle \cdot, \cdot \rangle_\Omega := \langle \cdot, \cdot \rangle_{\mathbb{L}^2(\Omega, \mathbb{R}^3)} \quad \text{and} \quad \|\cdot\|_\Omega := \|\cdot\|_{\mathbb{L}^2(\Omega, \mathbb{R}^3)}.$$

In order to define a weak solution of MLLG equations, we assume that the given functions  $\mathbf{m}_0$  and  $\mathbf{H}_0$  satisfy

$$\mathbf{m}_0 \in \mathbb{H}^1(D, \mathbb{R}^3), \quad |\mathbf{m}_0| = 1 \text{ a.e. in } D \quad \text{and} \quad \mathbf{H}_0 \in \mathbb{H}(\text{curl}; \tilde{D}). \quad (2.1)$$

For physical reasons (see [10]), these initial fields must satisfy

$$\text{div}(\mathbf{H}_0 + \chi_D \mathbf{m}_0) = 0 \text{ in } \tilde{D} \quad \text{and} \quad (\mathbf{H}_0 + \chi_D \mathbf{m}_0) \cdot \mathbf{n} = 0 \text{ on } \partial \tilde{D}. \quad (2.2)$$

Since equations (1.1) and (1.5) are equivalent (a proof of which can be found in [13]), instead of solving (1.1)–(1.4) we solve (1.2)–(1.5). A weak solution of the problem is defined in the following definition.

**Definition 2.1.** *Let the initial data  $(\mathbf{m}_0, \mathbf{H}_0)$  satisfy (2.1) and (2.2). Then  $(\mathbf{m}, \mathbf{H})$  is call a weak solution to (1.2)–(1.5) if, for all  $T > 0$ , there hold*

1.  $\mathbf{m} \in \mathbb{H}^1(D_T, \mathbb{R}^3)$  and  $|\mathbf{m}| = 1$  a.e. in  $D_T$ ;
2.  $\mathbf{H}, \mathbf{H}_t, \nabla \times \mathbf{H} \in \mathbb{L}^2(\tilde{D}_T, \mathbb{R}^3)$ ;
3. for all  $\phi \in \mathbb{C}^\infty(D_T)$  and  $\zeta \in \mathbb{C}^\infty(\tilde{D}_T)$ ,

$$\lambda_1 \langle \mathbf{m}_t, \phi \rangle_{D_T} + \lambda_2 \langle \mathbf{m} \times \mathbf{m}_t, \phi \rangle_{D_T} = \mu \langle \nabla \mathbf{m}, \nabla(\mathbf{m} \times \phi) \rangle_{D_T} + \mu \langle \mathbf{m} \times \mathbf{H}, \phi \rangle_{D_T} \quad (2.3)$$

and

$$\mu_0 \langle \mathbf{H}_t, \zeta \rangle_{\tilde{D}_T} + \sigma \langle \nabla \times \mathbf{H}, \nabla \times \zeta \rangle_{\tilde{D}_T} = -\mu_0 \langle \widetilde{\mathbf{m}}_t, \zeta \rangle_{\tilde{D}_T}, \quad (2.4)$$

where  $\mu = \lambda_1^2 + \lambda_2^2$ ;

4. in the sense of traces there holds

$$\mathbf{m}(0, \cdot) = \mathbf{m}_0, \quad (2.5)$$

5. for almost all  $T' \in (0, T)$ ,

$$\mathcal{E}(T') + \lambda_2 \mu^{-1} \|\mathbf{m}_t\|_{D_{T'}}^2 + \lambda_2 \mu^{-1} \|\mathbf{H}_t\|_{\tilde{D}_{T'}}^2 + 2\mu_0^{-1} \sigma \|\nabla \times \mathbf{H}\|_{\tilde{D}_{T'}}^2 \leq \mathcal{E}(0), \quad (2.6)$$

where

$$\mathcal{E}(T') = \|\nabla \mathbf{m}(T')\|_D^2 + \|\mathbf{H}(T')\|_{\tilde{D}}^2 + \lambda_2 \mu^{-1} \mu_0^{-1} \sigma \|\nabla \times \mathbf{H}(T')\|_{\tilde{D}}^2.$$

We next introduce the  $\theta$ -linear finite element scheme which approximates a weak solution  $(\mathbf{m}, \mathbf{H})$  defined in Definition 2.1.

Let  $\mathbb{T}_h$  be a regular tetrahedrization of the domain  $\tilde{D}$  into tetrahedra of maximal mesh-size  $h$ , and let  $\mathbb{T}_h|_D$  be its restriction to  $D \subset \tilde{D}$ . We denote by  $\mathcal{N}_h := \{\mathbf{x}_1, \dots, \mathbf{x}_N\}$  the set of vertices and by  $\mathcal{M}_h := \{e_1, \dots, e_M\}$  the set of edges.

To discretize the LLG equation (2.3), we introduce the finite element space  $\mathbb{V}_h \subset \mathbb{H}^1(D, \mathbb{R}^3)$  which is the space of all continuous piecewise linear functions on  $\mathbb{T}_h|_D$ . A basis for  $\mathbb{V}_h$  can be chosen to be  $(\phi_n)_{1 \leq n \leq N}$ , where  $\phi_n(\mathbf{x}_m) = \delta_{n,m}$ . Here  $\delta_{n,m}$  stands for the Kronecker symbol. The interpolation operator from  $\mathbb{C}^0(D, \mathbb{R}^3)$  onto  $\mathbb{V}_h$  is denoted by  $I_{\mathbb{V}_h}$ ,

$$I_{\mathbb{V}_h}(\mathbf{v}) = \sum_{n=1}^N \mathbf{v}(\mathbf{x}_n) \phi_n(\mathbf{x}) \quad \forall \mathbf{v} \in \mathbb{C}^0(D, \mathbb{R}^3).$$

To discretize Maxwell's equation (2.4), we use the space  $\mathbb{Y}_h$  of lowest order edge elements of Nedelec's first family [14]. It is known [14] that  $\mathbb{Y}_h$  is a subspace of  $\mathbb{H}(\text{curl}; \tilde{D})$  and that the set  $\{\boldsymbol{\psi}_1, \dots, \boldsymbol{\psi}_M\}$  is a basis for  $\mathbb{Y}_h$  if it satisfies

$$\begin{aligned} \boldsymbol{\psi}_q &\in \{\boldsymbol{\psi} : \tilde{D} \rightarrow \mathbb{R}^3 \mid \boldsymbol{\psi}|_K(\mathbf{x}) = \mathbf{a}_K + \mathbf{b}_K \times \mathbf{x}, \mathbf{a}_K, \mathbf{b}_K \in \mathbb{R}^3, \forall K \in \mathbb{T}_h\}, \\ \int_{e_p} \boldsymbol{\psi}_q \cdot \boldsymbol{\tau}_p ds &= \delta_{qp}, \end{aligned} \quad (2.7)$$

where  $\boldsymbol{\tau}_p$  is the unit vector in the direction of edge  $e_p$ . We also define the following interpolation operator  $I_{\mathbb{Y}_h}$  from  $\mathbb{C}^\infty(\tilde{D})$  onto  $\mathbb{Y}_h$ ,

$$I_{\mathbb{Y}_h}(\mathbf{u}) = \sum_{q=1}^M u_q \boldsymbol{\psi}_q \quad \forall \mathbf{u} \in \mathbb{C}^\infty(\tilde{D}, \mathbb{R}^3),$$

where  $u_q = \int_{e_q} \mathbf{u} \cdot \boldsymbol{\tau}_q ds$ .

Fixing a positive integer  $J$ , we choose the time step  $k$  to be  $k = T/J$  and define  $t_j = jk$ ,  $j = 0, \dots, J$ . For  $j = 1, 2, \dots, J$ , the functions  $\mathbf{m}(t_j, \cdot)$  and  $\mathbf{H}(t_j, \cdot)$  are approximated by  $\mathbf{m}_h^{(j)} \in \mathbb{V}_h$  and  $\mathbf{H}_h^{(j)} \in \mathbb{Y}_h$ , respectively.

We define the space  $\mathbb{W}_h^{(j)}$  by

$$\mathbb{W}_h^{(j)} := \left\{ \mathbf{w} \in \mathbb{V}_h \mid \mathbf{w}(\mathbf{x}_n) \cdot \mathbf{m}_h^{(j)}(\mathbf{x}_n) = 0, \ n = 1, \dots, N \right\},$$

and denote

$$\mathbf{H}_h^{(j+1/2)} := \frac{\mathbf{H}_h^{(j+1)} + \mathbf{H}_h^{(j)}}{2} \quad \text{and} \quad d_t \mathbf{H}_h^{(j+1)} := k^{-1}(\mathbf{H}_h^{(j+1)} - \mathbf{H}_h^{(j)}).$$

**Algorithm 2.1.**

**Step 1:** Set  $j = 0$ . Choose  $\mathbf{m}_h^{(0)} = I_{\mathbb{V}_h} \mathbf{m}_0$  and  $\mathbf{H}_h^{(0)} = I_{\mathbb{Y}_h} \mathbf{H}_0$ .

**Step 2:** Find  $(\mathbf{v}_h^{(j+1)}, \mathbf{H}_h^{(j+1)}) \in \mathbb{W}_h^{(j)} \times \mathbb{Y}_h$  satisfying

$$\begin{aligned} & \lambda_2 \left\langle \mathbf{v}_h^{(j+1)}, \mathbf{w}_h^{(j)} \right\rangle_D - \lambda_1 \left\langle \mathbf{m}_h^{(j)} \times \mathbf{v}_h^{(j+1)}, \mathbf{w}_h^{(j)} \right\rangle_D \\ & = -\mu \left\langle \nabla(\mathbf{m}_h^{(j)} + k\theta \mathbf{v}_h^{(j+1)}), \nabla \mathbf{w}_h^{(j)} \right\rangle_D + \mu \left\langle \mathbf{H}_h^{(j+1/2)}, \mathbf{w}_h^{(j)} \right\rangle_D \quad \forall \mathbf{w}_h^{(j)} \in \mathbb{W}_h^{(j)}, \end{aligned} \quad (2.8)$$

and

$$\begin{aligned} & \mu_0 \left\langle d_t \mathbf{H}_h^{(j+1)}, \boldsymbol{\zeta}_h \right\rangle_{\tilde{D}} + \sigma \left\langle \nabla \times \mathbf{H}_h^{(j+1/2)}, \nabla \times \boldsymbol{\zeta}_h \right\rangle_{\tilde{D}} \\ & = -\mu_0 \left\langle \mathbf{v}_h^{(j+1)}, \boldsymbol{\zeta}_h \right\rangle_{\tilde{D}} \quad \forall \boldsymbol{\zeta}_h \in \mathbb{Y}_h. \end{aligned} \quad (2.9)$$

**Step 3:** Define

$$\mathbf{m}_h^{(j+1)}(\mathbf{x}) := \sum_{n=1}^N \frac{\mathbf{m}_h^{(j)}(\mathbf{x}_n) + k\mathbf{v}_h^{(j+1)}(\mathbf{x}_n)}{\left| \mathbf{m}_h^{(j)}(\mathbf{x}_n) + k\mathbf{v}_h^{(j+1)}(\mathbf{x}_n) \right|} \phi_n(\mathbf{x}).$$

**Step 4:** Set  $j = j + 1$ , and return to Step 2.1 if  $j < J$ . Stop if  $j = J$ .

The parameter  $\theta$  in (2.8) can be chosen arbitrarily in  $[0, 1]$ . The method is explicit when  $\theta = 0$  and fully implicit when  $\theta = 1$ .

By the Lax–Milgram Theorem, for each  $j > 0$  there exists a unique solution  $(\mathbf{v}_h^{(j+1)}, \mathbf{H}_h^{(j+1)}) \in \mathbb{W}_h^{(j)} \times \mathbb{Y}_h$  of equations (2.8)–(2.9). Since  $\left| \mathbf{m}_h^{(j)}(\mathbf{x}_n) \right| = 1$  and  $\mathbf{v}_h^{(j+1)}(\mathbf{x}_n) \cdot \mathbf{m}_h^{(j)}(\mathbf{x}_n) = 0$  for all  $n = 1, \dots, N$ , there holds

$$\left| \mathbf{m}_h^{(j)}(\mathbf{x}_n) + k\mathbf{v}_h^{(j+1)}(\mathbf{x}_n) \right| \geq 1. \quad (2.10)$$

Therefore, the algorithm is well defined. There also holds  $\left| \mathbf{m}_h^{(j+1)}(\mathbf{x}_n) \right| = 1$  for  $n = 1, \dots, N$ .

### 3 Some technical lemmas

In this section we present some lemmas which will be used in the rest of the paper. We start by recalling the following lemma proved in [3].

**Lemma 3.1.** *If there holds*

$$\int_D \nabla \phi_i \cdot \nabla \phi_j d\mathbf{x} \leq 0 \quad \text{for all } i, j \in \{1, 2, \dots, J\} \text{ and } i \neq j, \quad (3.1)$$

*then for all  $\mathbf{u} \in \mathbb{V}_h$  satisfying  $|\mathbf{u}(\mathbf{x}_l)| \geq 1$ ,  $l = 1, 2, \dots, J$ , there holds*

$$\int_D \left| \nabla I_h \left( \frac{\mathbf{u}}{|\mathbf{u}|} \right) \right|^2 d\mathbf{x} \leq \int_D |\nabla \mathbf{u}|^2 d\mathbf{x}. \quad (3.2)$$

Condition (3.1) holds if all dihedral angles of the tetrahedra in  $\mathbb{T}_h|_D$  are less than or equal to  $\pi/2$ ; see [3]. In the sequel we assume that (3.1) holds.

The next lemma defines a discrete  $\mathbb{L}^p$ -norm in  $\mathbb{V}_h$  which is equivalent to the usual  $\mathbb{L}^p$ -norm.

**Lemma 3.2.** *There exist  $h$ -independent positive constants  $C_1$  and  $C_2$  such that for all  $p \in [1, \infty]$  and  $\mathbf{u} \in \mathbb{V}_h$  there holds*

$$C_1 \|\mathbf{u}\|_{\mathbb{L}^p(\Omega)}^p \leq h^d \sum_{n=1}^N |\mathbf{u}(\mathbf{x}_n)|^p \leq C_2 \|\mathbf{u}\|_{\mathbb{L}^p(\Omega)}^p,$$

where  $\Omega \subset \mathbb{R}^d$ ,  $d=1,2,3$ .

A proof of this lemma for  $p = 2$  and  $d = 2$  can be found in [11, Lemma 7.3] or [6, Lemma 1.12]. The result for general values of  $p$  and  $d$  can be obtained in the same manner.

The following lemma can be proved by using the technique in [11, Lemma 7.3].

**Lemma 3.3.** *There exists an  $h$ -independent positive constant  $C$  such that for each tetrahedron  $K \in \mathbb{T}_h$  and  $\mathbf{v} \in \mathbb{V}_h$  there holds*

$$||\mathbf{v}(\mathbf{x})| - |\mathbf{v}(\mathbf{x}_i)|| \leq Ch |\nabla \mathbf{v}(\mathbf{x})| \quad \text{for all } \mathbf{x} \in K,$$

where  $\{\mathbf{x}_i\}_{i=1,2,3}$  are the vertices of  $K$ .

Finally the following lemma is elementary; the proof of which is included for completeness.

**Lemma 3.4.** *The solutions  $(\mathbf{m}_h^{(j)}, \mathbf{v}_h^{(j+1)})$ ,  $j = 0, 1, \dots, J$ , obtained from Algorithm 2.1 satisfy*

$$\left| \frac{\mathbf{m}_h^{(j+1)}(\mathbf{x}_n) - \mathbf{m}_h^{(j)}(\mathbf{x}_n)}{k} \right| \leq |\mathbf{v}_h^{(j+1)}(\mathbf{x}_n)| \quad \forall n = 1, 2, \dots, N, \quad j = 0, \dots, J. \quad (3.3)$$

*Proof.* By using the definition of  $\mathbf{m}_h^{(j+1)}$ , the property  $\mathbf{m}_h^{(j)}(\mathbf{x}_n) \cdot \mathbf{v}_h^{(j+1)}(\mathbf{x}_n) = 0$ , and the identity

$$|\mathbf{m}_h^{(j)}(\mathbf{x}_n) + k\mathbf{v}_h^{(j+1)}(\mathbf{x}_n)| = \sqrt{1 + k^2|\mathbf{v}_h^{(j+1)}(\mathbf{x}_n)|^2}$$

we obtain

$$\begin{aligned} \left| \frac{\mathbf{m}_h^{(j+1)}(\mathbf{x}_n) - \mathbf{m}_h^{(j)}(\mathbf{x}_n)}{k} \right|^2 &= \left| \frac{\mathbf{m}_h^{(j)}(\mathbf{x}_n) + k\mathbf{v}_h^{(j+1)}(\mathbf{x}_n)}{k|\mathbf{m}_h^{(j)}(\mathbf{x}_n) + k\mathbf{v}_h^{(j+1)}(\mathbf{x}_n)|} - \frac{\mathbf{m}_h^{(j)}(\mathbf{x}_n)}{k} \right|^2 \\ &= \frac{|\mathbf{m}_h^{(j)}(\mathbf{x}_n)(1 - |\mathbf{m}_h^{(j)}(\mathbf{x}_n) + k\mathbf{v}_h^{(j+1)}(\mathbf{x}_n)|) + k\mathbf{v}_h^{(j+1)}(\mathbf{x}_n)|^2}{k^2|\mathbf{m}_h^{(j)}(\mathbf{x}_n) + k\mathbf{v}_h^{(j+1)}(\mathbf{x}_n)|^2} \\ &= \frac{2 + 2k^2|\mathbf{v}_h^{(j+1)}(\mathbf{x}_n)|^2 - 2\sqrt{1 + k^2|\mathbf{v}_h^{(j+1)}(\mathbf{x}_n)|^2}}{k^2(1 + k^2|\mathbf{v}_h^{(j+1)}(\mathbf{x}_n)|^2)} \\ &= 2 \frac{\sqrt{1 + k^2|\mathbf{v}_h^{(j+1)}(\mathbf{x}_n)|^2} - 1}{k^2\sqrt{1 + k^2|\mathbf{v}_h^{(j+1)}(\mathbf{x}_n)|^2}}. \end{aligned}$$

Using the fact that

$$2 \leq \sqrt{1 + k^2|\mathbf{v}_h^{(j+1)}(\mathbf{x}_n)|^2} + 1$$

we deduce

$$\begin{aligned} \left| \frac{\mathbf{m}_h^{(j+1)}(\mathbf{x}_n) - \mathbf{m}_h^{(j)}(\mathbf{x}_n)}{k} \right|^2 &\leq \frac{\left( \sqrt{1 + k^2|\mathbf{v}_h^{(j+1)}(\mathbf{x}_n)|^2} + 1 \right) \left( \sqrt{1 + k^2|\mathbf{v}_h^{(j+1)}(\mathbf{x}_n)|^2} - 1 \right)}{k^2} \\ &= |\mathbf{v}_h^{(j+1)}(\mathbf{x}_n)|^2, \end{aligned}$$

proving the lemma.  $\square$

In the following section, we show that our numerical solution converges to a weak solution of the problem (1.2)–(1.5).

## 4 Existence of weak solutions

The next lemma provides a bound in the  $\mathbb{L}^2$ -norm for the discrete solutions.

**Lemma 4.1.** *The sequence  $\left\{(\mathbf{m}_h^{(j)}, \mathbf{v}_h^{(j+1)}, \mathbf{H}_h^{(j)})\right\}_{j=0,1,\dots,J}$  produced by Algorithm 2.1 satisfies*

$$\begin{aligned} \mathcal{E}_h^{(j)} + C \sum_{i=0}^{j-1} k \|\mathbf{v}_h^{(i+1)}\|_D^2 + \lambda_2 \mu^{-1} \sum_{i=0}^{j-1} k \|d_t \mathbf{H}_h^{(i+1)}\|_{\tilde{D}}^2 \\ + 2\mu_0^{-1} \sigma \sum_{i=0}^{j-1} k \|\nabla \times \mathbf{H}_h^{(i+1/2)}\|_{\tilde{D}}^2 \leq \mathcal{E}_h^0, \end{aligned} \quad (4.1)$$

where

$$\mathcal{E}_h^{(j)} = \|\nabla \mathbf{m}_h^{(j)}\|_D^2 + \|\mathbf{H}_h^{(j)}\|_{\tilde{D}}^2 + \lambda_2 \mu^{-1} \mu_0^{-1} \sigma \|\nabla \times \mathbf{H}_h^{(j)}\|_{\tilde{D}}^2,$$

and

$$C = \begin{cases} \lambda_2 \mu^{-1}, & \theta \in [\frac{1}{2}, 1] \\ \lambda_2 \mu^{-1} - (1 - 2\theta) C_1 k h^{-2}, & \theta \in [0, \frac{1}{2}), \end{cases}$$

in which  $C_1$  is a positive constant which is independent with  $j$ ,  $k$  and  $h$ .

*Proof.* Choosing  $\mathbf{w}_h^{(j)} = \mathbf{v}_h^{(j+1)}$  in (2.8) and  $\boldsymbol{\zeta}_h = \mathbf{H}_h^{(j+1/2)}$  in (2.9), we obtain

$$\lambda_2 \|\mathbf{v}_h^{(j+1)}\|_D^2 + k\theta\mu \|\nabla \mathbf{v}_h^{(j+1)}\|_D^2 = -\mu \left\langle \nabla \mathbf{m}_h^{(j)}, \nabla \mathbf{v}_h^{(j+1)} \right\rangle_D + \mu \left\langle \mathbf{H}_h^{(j+1/2)}, \mathbf{v}_h^{(j+1)} \right\rangle_D \quad (4.2)$$

$$\frac{\mu_0}{2} d_t \|\mathbf{H}_h^{(j+1)}\|_{\tilde{D}}^2 + \sigma \|\nabla \times \mathbf{H}_h^{(j+1/2)}\|_{\tilde{D}}^2 = -\mu_0 \left\langle \mathbf{v}_h^{(j+1)}, \mathbf{H}_h^{(j+1/2)} \right\rangle_{\tilde{D}}. \quad (4.3)$$

Multiplying  $\mu\mu_0^{-1}$  to both sides of (4.3) and adding the resulting equation to (4.2), we deduce

$$\begin{aligned} \lambda_2 \|\mathbf{v}_h^{(j+1)}\|_D^2 + k\theta\mu \|\nabla \mathbf{v}_h^{(j+1)}\|_D^2 + \frac{\mu}{2} d_t \|\mathbf{H}_h^{(j+1)}\|_{\tilde{D}}^2 \\ + \mu\mu_0^{-1} \sigma \|\nabla \times \mathbf{H}_h^{(j+1/2)}\|_{\tilde{D}}^2 = -\mu \left\langle \nabla \mathbf{m}_h^{(j)}, \nabla \mathbf{v}_h^{(j+1)} \right\rangle_D. \end{aligned} \quad (4.4)$$

Since  $\mathbf{m}_h^{(j)} + k\mathbf{v}_h^{(j+1)} \in \mathbb{V}_h$  and

$$\mathbf{m}_h^{(j+1)} = I_h \left( \frac{\mathbf{m}_h^{(j)} + k\mathbf{v}_h^{(j+1)}}{|\mathbf{m}_h^{(j)} + k\mathbf{v}_h^{(j+1)}|} \right),$$

it follows from (2.10) and Lemma 3.1 that

$$\|\nabla \mathbf{m}_h^{(j+1)}\|_D^2 \leq \|\nabla (\mathbf{m}_h^{(j)} + k\mathbf{v}_h^{(j+1)})\|_D^2.$$

Equivalently, we have

$$\|\nabla \mathbf{m}_h^{(j+1)}\|_D^2 \leq \|\nabla \mathbf{m}_h^{(j)}\|_D^2 + k^2 \|\nabla \mathbf{v}_h^{(j+1)}\|_D^2 + 2k \left\langle \nabla \mathbf{m}_h^{(j)}, \nabla \mathbf{v}_h^{(j+1)} \right\rangle_D. \quad (4.5)$$



Equality (4.4) is used to obtain from (4.5) the following inequality

$$\begin{aligned} \|\nabla \mathbf{m}_h^{(j+1)}\|_D^2 &\leq \|\nabla \mathbf{m}_h^{(j)}\|_D^2 - k^2(2\theta - 1)\|\nabla \mathbf{v}_h^{(j+1)}\|_D^2 - 2k\lambda_2\mu^{-1}\|\mathbf{v}_h^{(j+1)}\|_D^2 \\ &\quad - kd_t\|\mathbf{H}_h^{(j+1)}\|_{\tilde{D}}^2 - 2k\mu_0^{-1}\sigma\|\nabla \times \mathbf{H}_h^{(j+1/2)}\|_{\tilde{D}}^2. \end{aligned}$$

Hence,

$$\begin{aligned} \|\nabla \mathbf{m}_h^{(j+1)}\|_D^2 + \|\mathbf{H}_h^{(j+1)}\|_{\tilde{D}}^2 + 2k\lambda_2\mu^{-1}\|\mathbf{v}_h^{(j+1)}\|_D^2 + 2k\mu_0^{-1}\sigma\|\nabla \times \mathbf{H}_h^{(j+1/2)}\|_{\tilde{D}}^2 \\ + k^2(2\theta - 1)\|\nabla \mathbf{v}_h^{(j+1)}\|_D^2 \leq \|\nabla \mathbf{m}_h^{(j)}\|_D^2 + \|\mathbf{H}_h^{(j)}\|_{\tilde{D}}^2. \end{aligned} \quad (4.6)$$

Next choosing  $\zeta_h = d_t \mathbf{H}_h^{(j+1)}$  in equation (2.9), we obtain

$$\begin{aligned} 2k\mu_0\|d_t \mathbf{H}_h^{(j+1)}\|_{\tilde{D}}^2 + \sigma\|\nabla \times \mathbf{H}_h^{(j+1)}\|_{\tilde{D}}^2 &= \sigma\|\nabla \times \mathbf{H}_h^{(j)}\|_{\tilde{D}}^2 \\ &\quad - 2k\mu_0\left\langle \mathbf{v}_h^{(j+1)}, d_t \mathbf{H}_h^{(j+1)} \right\rangle_{\tilde{D}}. \end{aligned}$$

The term  $-2k\mu_0\left\langle \mathbf{v}_h^{(j+1)}, d_t \mathbf{H}_h^{(j+1)} \right\rangle_{\tilde{D}}$  can be estimated by

$$-2k\mu_0\left\langle \mathbf{v}_h^{(j+1)}, d_t \mathbf{H}_h^{(j+1)} \right\rangle_{\tilde{D}} \leq k\mu_0\|\mathbf{v}_h^{(j+1)}\|_D^2 + k\mu_0\|d_t \mathbf{H}_h^{(j+1)}\|_{\tilde{D}}^2.$$

Therefore, we deduce

$$k\mu_0\|d_t \mathbf{H}_h^{(j+1)}\|_{\tilde{D}}^2 + \sigma\|\nabla \times \mathbf{H}_h^{(j+1)}\|_{\tilde{D}}^2 \leq \sigma\|\nabla \times \mathbf{H}_h^{(j)}\|_{\tilde{D}}^2 + k\mu_0\|\mathbf{v}_h^{(j+1)}\|_D^2. \quad (4.7)$$

Multiplying  $\lambda_2\mu^{-1}\mu_0^{-1}$  to both sides of (4.7) and adding the resulting equation to (4.6), we obtain

$$\begin{aligned} &\|\nabla \mathbf{m}_h^{(j+1)}\|_D^2 + \|\mathbf{H}_h^{(j+1)}\|_{\tilde{D}}^2 + \lambda_2\mu^{-1}\mu_0^{-1}\sigma\|\nabla \times \mathbf{H}_h^{(j+1)}\|_{\tilde{D}}^2 \\ &\quad + k\lambda_2\mu^{-1}\|\mathbf{v}_h^{(j+1)}\|_D^2 + k\lambda_2\mu^{-1}\|d_t \mathbf{H}_h^{(j+1)}\|_{\tilde{D}}^2 + 2k\mu_0^{-1}\sigma\|\nabla \times \mathbf{H}_h^{(j+1/2)}\|_{\tilde{D}}^2 \\ &\quad + k^2(2\theta - 1)\|\nabla \mathbf{v}_h^{(j+1)}\|_D^2 \\ &\leq \|\nabla \mathbf{m}_h^{(j)}\|_D^2 + \|\mathbf{H}_h^{(j)}\|_{\tilde{D}}^2 + \lambda_2\mu^{-1}\mu_0^{-1}\sigma\|\nabla \times \mathbf{H}_h^{(j)}\|_{\tilde{D}}^2. \end{aligned}$$

Replacing  $j$  by  $i$  in the above inequality and summing over  $i$  from 0 to  $j-1$  yield

$$\begin{aligned} &\|\nabla \mathbf{m}_h^{(j)}\|_D^2 + \|\mathbf{H}_h^{(j)}\|_{\tilde{D}}^2 + \lambda_2\mu^{-1}\mu_0^{-1}\sigma\|\nabla \times \mathbf{H}_h^{(j)}\|_{\tilde{D}}^2 + \lambda_2\mu^{-1}\sum_{i=0}^{j-1}k\|\mathbf{v}_h^{(i+1)}\|_D^2 \\ &\quad + \lambda_2\mu^{-1}\sum_{i=0}^{j-1}k\|d_t \mathbf{H}_h^{(i+1)}\|_{\tilde{D}}^2 + 2\mu_0^{-1}\sigma\sum_{i=0}^{j-1}k\|\nabla \times \mathbf{H}_h^{(i+1/2)}\|_{\tilde{D}}^2 \\ &\quad + k^2(2\theta - 1)\sum_{i=0}^{j-1}\|\nabla \mathbf{v}_h^{(i+1)}\|_D^2 \\ &\leq \|\nabla \mathbf{m}_h^0\|_D^2 + \|\mathbf{H}_h^0\|_{\tilde{D}}^2 + \lambda_2\mu^{-1}\mu_0^{-1}\sigma\|\nabla \times \mathbf{H}_h^0\|_{\tilde{D}}^2. \end{aligned} \quad (4.8)$$

When  $\theta \in [\frac{1}{2}, 1]$ , the term  $k^2(2\theta - 1) \sum_{i=0}^{j-1} \|\nabla \mathbf{v}_h^{(i+1)}\|_D^2$  is nonnegative. Hence, from inequality (4.8) we obtain (4.1) where  $C = \lambda_2 \mu^{-1}$ . When  $\theta \in [0, \frac{1}{2})$ , using the inverse estimate we obtain

$$C_1 k^2 h^{-2} (2\theta - 1) \sum_{i=0}^{j-1} \|\mathbf{v}_h^{(i+1)}\|_D^2 \leq k^2 (2\theta - 1) \sum_{i=0}^{j-1} \|\nabla \mathbf{v}_h^{(i+1)}\|_D^2, \quad (4.9)$$

where,  $C_1$  is a positive constant which is independent with  $j$ ,  $k$  and  $h$ . Hence, from inequality (4.8) we obtain (4.1) where  $C = \lambda_2 \mu^{-1} - C_1 k h^{-2} (1 - 2\theta)$ . This completes the proof of the lemma.  $\square$

**Remark 4.2.** *The constant  $C$  in the above lemma is positive when  $\theta \in [1/2, 1]$ . When  $\theta \in [0, 1/2)$  the additional condition  $k = o(h^2)$  assures us that  $C$  is positive when  $h$  and  $k$  are sufficiently small. This condition will be required later in the following lemma and theorem.*

The discrete solutions  $\mathbf{m}_h^{(j)}$ ,  $\mathbf{v}_h^{(j+1)}$  and  $\mathbf{H}_h^{(j)}$  constructed via Algorithm 2.1 are interpolated in time in the following definition.

**Definition 4.3.** *For each  $t \in [0, T]$ , let  $j \in \{0, \dots, J\}$  be such that  $t \in [t_j, t_{j+1})$ . We define for  $t \in [0, T]$  and  $\mathbf{x} \in D$*

$$\begin{aligned} \mathbf{m}_{h,k}(t, \mathbf{x}) &:= \frac{t - t_j}{k} \mathbf{m}_h^{(j+1)}(\mathbf{x}) + \frac{t_{j+1} - t}{k} \mathbf{m}_h^{(j)}(\mathbf{x}), \\ \mathbf{m}_{h,k}^-(t, \mathbf{x}) &:= \mathbf{m}_h^{(j)}(\mathbf{x}), \\ \mathbf{v}_{h,k}(t, \mathbf{x}) &:= \mathbf{v}_h^{(j+1)}(\mathbf{x}), \\ \mathbf{H}_{h,k}(t, \mathbf{x}) &:= \frac{t - t_j}{k} \mathbf{H}_h^{(j+1)}(\mathbf{x}) + \frac{t_{j+1} - t}{k} \mathbf{H}_h^{(j)}(\mathbf{x}), \\ \widetilde{\mathbf{H}}_{h,k}(t, \mathbf{x}) &:= \frac{1}{2} \left( \mathbf{H}_h^{(j+1)}(\mathbf{x}) + \mathbf{H}_h^{(j)}(\mathbf{x}) \right), \\ \text{and } \mathbf{H}_{h,k}^-(t, \mathbf{x}) &:= \mathbf{H}_h^{(j)}(\mathbf{x}). \end{aligned}$$

The following lemma shows that  $\{\mathbf{m}_{h,k}\}$ ,  $\{\mathbf{m}_{h,k}^-\}$  and  $\{\mathbf{v}_{h,k}\}$  converge (up to the extraction of subsequences) as  $h$  and  $k$  tend to 0.

**Lemma 4.4.** *Assume that  $h$  and  $k$  go to 0 with a further condition  $k = o(h^2)$  when  $\theta \in [0, \frac{1}{2})$  and no condition otherwise. There exist  $\mathbf{m} \in \mathbb{H}^1(D_T, \mathbb{R}^3)$  and*

$\mathbf{H} \in \mathbb{H}^1(0, T, \mathbb{L}^2(\tilde{D}))$  such that  $\nabla \times \mathbf{H}$  belongs to  $\mathbb{L}^2(\tilde{D}_T)$  and

$$\mathbf{m}_{h,k} \rightarrow \mathbf{m} \text{ strongly in } \mathbb{L}^2(D_T), \quad (4.10)$$

$$\frac{\partial \mathbf{m}_{h,k}}{\partial t} \rightharpoonup \mathbf{m}_t \text{ weakly in } \mathbb{L}^2(D_T), \quad (4.11)$$

$$\mathbf{v}_{h,k} \rightharpoonup \mathbf{m}_t \text{ weakly in } \mathbb{L}^2(D_T), \quad (4.12)$$

$$\mathbf{m}_{h,k}^- \rightarrow \mathbf{m} \text{ strongly in } \mathbb{L}^2(D_T), \quad (4.13)$$

$$|\mathbf{m}| = 1 \text{ a.e. in } D_T, \quad (4.14)$$

$$\mathbf{H}_{h,k} \rightharpoonup \mathbf{H} \text{ weakly in } \mathbb{H}^1(0, T, \mathbb{L}^2(\tilde{D})), \quad (4.15)$$

$$\nabla \times \tilde{\mathbf{H}}_{h,k} \rightharpoonup \nabla \times \mathbf{H} \text{ weakly in } \mathbb{L}^2(\tilde{D}_T), \quad (4.16)$$

$$\text{and } \nabla \times \mathbf{H}_{h,k}^- \rightharpoonup \nabla \times \mathbf{H} \text{ weakly in } \mathbb{L}^2(\tilde{D}_T). \quad (4.17)$$

*Proof.*

Proof of (4.10) and (4.11):

Our goal is to prove that  $\{\mathbf{m}_{h,k}\}$  is bounded in  $\mathbb{H}^1(D_T, \mathbb{R}^3)$  and then use the Banach–Alaoglu Theorem. We note from Definition 4.3 that it suffices to prove that

$$\|\mathbf{m}_h^{(j)}\|_D \leq c, \quad (4.18)$$

$$\|\nabla \mathbf{m}_h^{(j)}\|_D \leq c, \quad (4.19)$$

$$k \sum_{j=0}^{J-1} \left\| \frac{\mathbf{m}_h^{(j+1)} - \mathbf{m}_h^{(j)}}{k} \right\|_D^2 \leq c, \quad (4.20)$$

where the generic constant  $c$  is independent of  $j$ ,  $h$ , and  $k$ . Indeed, it follows from Definition 4.3 and the Cauchy–Schwarz inequality that

$$\begin{aligned} \|\mathbf{m}_{h,k}\|_{D_T}^2 &\leq ck \sum_{j=0}^{J-1} \left( \|\mathbf{m}_h^{(j+1)}\|_D^2 + \|\mathbf{m}_h^{(j)}\|_D^2 \right), \\ \|\nabla \mathbf{m}_{h,k}\|_{D_T}^2 &\leq ck \sum_{j=0}^{J-1} \left( \|\nabla \mathbf{m}_h^{(j+1)}\|_D^2 + \|\nabla \mathbf{m}_h^{(j)}\|_D^2 \right), \\ \left\| \frac{\partial \mathbf{m}_{h,k}}{\partial t} \right\|_{D_T}^2 &= \sum_{j=0}^{J-1} k \left\| \frac{\mathbf{m}_h^{(j+1)} - \mathbf{m}_h^{(j)}}{k} \right\|_D^2. \end{aligned}$$

In order to prove (4.18) we note that for every  $\mathbf{x} \in D$  there are at most 4 basis functions  $\phi_{n_1}, \phi_{n_2}, \phi_{n_3}$  and  $\phi_{n_4}$  being nonzero at  $\mathbf{x}$ . This together with  $|\mathbf{m}_h^{(j)}(\mathbf{x}_{n_i})| = 1$  and  $\sum_{i=1}^4 \phi_{n_i}(\mathbf{x}) = 1$  yields

$$|\mathbf{m}_h^{(j)}(\mathbf{x})|^2 = \left| \sum_{i=1}^4 \mathbf{m}_h^{(j)}(\mathbf{x}_{n_i}) \phi_{n_i}(\mathbf{x}) \right|^2 \leq 1. \quad (4.21)$$

This implies (4.18) with a constant  $c = |D|^{1/2}$  where  $|D|$  is the measure of the domain  $D$ .

Inequality (4.19) is proved in Lemma 4.1. In order to prove inequality (4.20), we note that Lemma 3.4 and Lemma 3.2 imply

$$\left\| \frac{\mathbf{m}_h^{(j+1)} - \mathbf{m}_h^{(j)}}{k} \right\|_D \leq c \left\| \mathbf{v}_h^{(j+1)} \right\|_D.$$

By using this inequality, Lemma 4.1 and Remark 4.2 we deduce

$$k \sum_{j=0}^{J-1} \left\| \frac{\mathbf{m}_h^{(j+1)} - \mathbf{m}_h^{(j)}}{k} \right\|_D^2 \leq k \sum_{j=0}^{J-1} c \left\| \mathbf{v}_h^{(j+1)} \right\|_D^2 \leq c.$$

The Banach–Alaoglu Theorem implies the existence of a subsequence of  $\{\mathbf{m}_{h,k}\}$  which converges weakly to a function  $\mathbf{m} \in \mathbb{H}^1(D_T)$  as  $k$  and  $h$  tend to zero. This implies (4.10) and (4.11).

Proof of (4.12):

From (4.1) and Remark 4.2, it is straightforward to show that  $\{\mathbf{v}_{h,k}\}$  is bounded in  $\mathbb{L}^2(D_T)$ . Hence, there exists a subsequence of  $\{\mathbf{v}_{h,k}\}$  which converges weakly to a function  $\mathbf{v} \in \mathbb{L}^2(D_T)$ . The problem reduces to proving that  $\mathbf{m}_t$  equals  $\mathbf{v}$  in  $\mathbb{L}^2(D_T)$ . In order to show this we choose for each  $\boldsymbol{\psi} \in \mathbb{L}^2(D_T)$  a sequence  $\{\boldsymbol{\psi}_i\} \in \mathbb{C}_0^\infty(D_T)$  converging to  $\boldsymbol{\psi}$  in  $\mathbb{L}^2(D_T)$  as  $i$  tends to infinity. We then have

$$\begin{aligned} |\langle \mathbf{m}_t - \mathbf{v}, \boldsymbol{\psi} \rangle_{D_T}| &\leq |\langle \mathbf{m}_t - \mathbf{v}, \boldsymbol{\psi}_i - \boldsymbol{\psi} \rangle_{D_T}| + \left| \left\langle \mathbf{m}_t - \frac{\partial \mathbf{m}_{h,k}}{\partial t}, \boldsymbol{\psi}_i \right\rangle_{D_T} \right| \\ &\quad + \left| \left\langle \frac{\partial \mathbf{m}_{h,k}}{\partial t} - \mathbf{v}_{h,k}, \boldsymbol{\psi}_i \right\rangle_{D_T} \right| + |\langle \mathbf{v}_{h,k} - \mathbf{v}, \boldsymbol{\psi}_i \rangle_{D_T}| \\ &\leq \|\mathbf{m}_t - \mathbf{v}\|_{D_T} \|\boldsymbol{\psi}_i - \boldsymbol{\psi}\|_{D_T} + \left| \left\langle \mathbf{m}_t - \frac{\partial \mathbf{m}_{h,k}}{\partial t}, \boldsymbol{\psi}_i \right\rangle_{D_T} \right| \\ &\quad + \left\| \frac{\partial \mathbf{m}_{h,k}}{\partial t} - \mathbf{v}_{h,k} \right\|_{\mathbb{L}^1(D_T)} \|\boldsymbol{\psi}_i\|_{\mathbb{L}^\infty(D_T)} + |\langle \mathbf{v}_{h,k} - \mathbf{v}, \boldsymbol{\psi}_i \rangle_{D_T}| \\ &=: T_1 + \dots + T_4. \end{aligned} \tag{4.22}$$

By letting  $h, k \rightarrow 0$  and then  $i \rightarrow \infty$  we have  $T_i \rightarrow 0$  for  $i = 1, 2$  and  $4$ . It remains to show that  $T_3 \rightarrow 0$ .

It is clear from the definition of  $\mathbf{m}_h^{(j+1)}$  in Algorithm 2.1 that

$$\left| \mathbf{m}_h^{(j+1)}(\mathbf{x}_n) - \mathbf{m}_h^{(j)}(\mathbf{x}_n) - k \mathbf{v}_h^{(j+1)}(\mathbf{x}_n) \right| = \left| \mathbf{m}_h^{(j)}(\mathbf{x}_n) + k \mathbf{v}_h^{(j+1)}(\mathbf{x}_n) \right| - 1. \tag{4.23}$$

It easily follows from  $|\mathbf{m}_h^{(j)}(\mathbf{x}_n)| = 1$  and  $\mathbf{v}_h^{(j+1)}(\mathbf{x}_n) \cdot \mathbf{m}_h^{(j)}(\mathbf{x}_n) = 0$  that

$$\left| \mathbf{m}_h^{(j)}(\mathbf{x}_n) + k\mathbf{v}_h^{(j+1)}(\mathbf{x}_n) \right| \leq \frac{1}{2}k^2 \left| \mathbf{v}_h^{(j+1)}(\mathbf{x}_n) \right|^2 + 1.$$

The above inequality and (4.23) yield

$$\left| \frac{\mathbf{m}_h^{(j+1)}(\mathbf{x}_n) - \mathbf{m}_h^{(j)}(\mathbf{x}_n)}{k} - \mathbf{v}_h^{(j+1)}(\mathbf{x}_n) \right| \leq \frac{1}{2}k \left| \mathbf{v}_h^{(j+1)}(\mathbf{x}_n) \right|^2.$$

By using Lemma 3.2 we deduce

$$\left\| \frac{\partial \mathbf{m}_{h,k}}{\partial t}(t) - \mathbf{v}_{h,k}(t) \right\|_{\mathbb{L}^1(D)} \leq ck \|\mathbf{v}_{h,k}(t)\|_D^2 \quad \text{for } t \in [t_j, t_{j+1}).$$

Integrating both sides of this inequality with respect to  $t$  over an interval  $[t_j, t_{j+1})$  and summing over  $j$  from 0 to  $J-1$  yield, noting the boundedness of  $\{\|\mathbf{v}_{h,k}\|_{D_T}\}$ ,

$$\left\| \frac{\partial \mathbf{m}_{h,k}}{\partial t} - \mathbf{v}_{h,k} \right\|_{\mathbb{L}^1(D_T)} \leq ck \|\mathbf{v}_{h,k}\|_{D_T}^2 \leq ck \rightarrow 0 \quad \text{as } h, k \rightarrow 0.$$

Thus  $T_3 \rightarrow 0$  as  $h, k \rightarrow 0$  and  $i \rightarrow \infty$ . It follows from (4.22) that

$$|\langle \mathbf{m}_t - \mathbf{v}, \boldsymbol{\psi} \rangle_{D_T}| = 0 \quad \forall \boldsymbol{\psi} \in \mathbb{L}^2(D_T).$$

This proves (4.12).

Proof of (4.13):

It is clear from the definition of  $\mathbf{m}_{h,k}^-$  and  $\mathbf{m}_{h,k}$  that for  $t \in [t_j, t_{j+1})$  there holds

$$\|\mathbf{m}_{h,k}(t) - \mathbf{m}_{h,k}^-(t)\|_D = \left\| (t - t_j) \frac{\mathbf{m}_h^{(j+1)} - \mathbf{m}_h^{(j)}}{k} \right\|_D \leq k \left\| \frac{\partial(\mathbf{m}_{h,k}(t, x))}{\partial t} \right\|_D.$$

Integrating both sides of this inequality with respect to  $t$  over an interval  $[t_j, t_{j+1})$  and summing over  $j$  from 0 to  $(J-1)$  yield

$$\|\mathbf{m}_{h,k} - \mathbf{m}_{h,k}^-\|_{D_T} \leq k \left\| \frac{\partial \mathbf{m}_{h,k}}{\partial t} \right\|_{D_T} \leq ck \rightarrow 0 \quad \text{as } h, k \rightarrow 0.$$

The above result and (4.10) imply (4.13).

Proof of (4.14):

Using Lemma 3.3 and noting that  $|\mathbf{m}_h^{(j)}(\mathbf{x}_n)| = 1$  for  $n = 1, \dots, N$ , we deduce

$$\left| |\mathbf{m}_h^{(j)}(\mathbf{x})| - 1 \right|^2 \leq Ch^2 \left| \nabla \mathbf{m}_h^{(j)}(\mathbf{x}) \right|^2 \quad \text{for all } \mathbf{x} \in D.$$

Integrating both sides of the above inequality on  $[t_j, t_{j+1}) \times D$ , using Lemma 4.1 and noting Remark 4.2, we obtain

$$\int_{t_j}^{t_{j+1}} \int_D \left| 1 - |\mathbf{m}_h^{(j)}(\mathbf{x})| \right|^2 d\mathbf{x} dt \leq ch^2 \int_{t_j}^{t_{j+1}} \left\| \nabla \mathbf{m}_h^{(j)} \right\|_D^2 \leq ckh^2.$$

Hence

$$\int_{D_T} |1 - |\mathbf{m}_{h,k}^-||^2 d\mathbf{x} dt \rightarrow 0 \text{ as } h, k \rightarrow 0.$$

We infer from (4.13) that

$$|\mathbf{m}| = 1 \text{ a.e. in } D_T.$$

Proof of (4.15), (4.16) and (4.17):

By using the same arguments as above, we obtain these results, completing the proof of the lemma.  $\square$

We are now able to prove the main result of this paper.

**Theorem 4.5.** *Assume that  $h$  and  $k$  go to 0 with the following conditions*

$$\begin{cases} k = o(h^2) & \text{when } 0 \leq \theta < 1/2, \\ k = o(h) & \text{when } \theta = 1/2, \\ \text{no condition} & \text{when } 1/2 < \theta \leq 1. \end{cases} \quad (4.24)$$

*Then the limits  $(\mathbf{m}, \mathbf{H})$  given by Lemma 4.4 is a weak solution of the MLLG equations (2.3)–(2.4).*

*Proof.* For any  $\phi \in \mathbb{C}^\infty(D_T)$ ,  $\zeta \in \mathbb{C}^\infty(\tilde{D}_T)$ , and  $t \in [t_j, t_{j+1})$ , we define

$$\mathbf{w}_{h,k}(t, \cdot) := I_{\mathbb{V}_h}(\mathbf{m}_{h,k}^- \times \phi(t, \cdot)) \quad \text{and} \quad \zeta_h(t, \cdot) := I_{\mathbb{V}_h}(\zeta(t, \cdot)).$$

In equations (2.8) and (2.9), replacing  $\mathbf{w}_h^{(j)}$  and  $\zeta_h$  by  $\mathbf{w}_{h,k}(t)$  and  $\zeta_h(t)$ , respectively, and using Definition 4.3, we rewrite (2.8)–(2.9) as

$$\begin{aligned} & -\lambda_1 \langle \mathbf{m}_{h,k}^-(t) \times \mathbf{v}_{h,k}(t), \mathbf{w}_{h,k}(t) \rangle_D + \lambda_2 \langle \mathbf{v}_{h,k}(t), \mathbf{w}_{h,k}(t) \rangle_D \\ & = -\mu \langle \nabla(\mathbf{m}_{h,k}^-(t) + k\theta \mathbf{v}_{h,k}(t)), \nabla \mathbf{w}_{h,k}(t) \rangle_D + \mu \langle \widetilde{\mathbf{H}}_{h,k}(t), \mathbf{w}_{h,k}(t) \rangle_D, \end{aligned}$$

and

$$\mu_0 \left\langle \frac{\partial \mathbf{H}_{h,k}}{\partial t}(t), \zeta_h(t) \right\rangle_{\tilde{D}} + \sigma \left\langle \nabla \times \widetilde{\mathbf{H}}_{h,k}(t), \nabla \times \zeta_h(t) \right\rangle_{\tilde{D}} = -\mu_0 \langle \mathbf{v}_{h,k}(t), \zeta_h(t) \rangle_{\tilde{D}}.$$

Integrating both sides of these equations with respect to  $t$  over an interval  $[t_j, t_{j+1})$  and summing over  $j$  from 0 to  $J-1$  yield

$$\begin{aligned} & -\lambda_1 \langle \mathbf{m}_{h,k}^- \times \mathbf{v}_{h,k}, \mathbf{w}_{h,k} \rangle_{D_T} + \lambda_2 \langle \mathbf{v}_{h,k}, \mathbf{w}_{h,k} \rangle_{D_T} \\ & = -\mu \langle \nabla(\mathbf{m}_{h,k}^- + k\theta \mathbf{v}_{h,k}), \nabla \mathbf{w}_{h,k} \rangle_{D_T} + \mu \langle \widetilde{\mathbf{H}}_{h,k}, \mathbf{w}_{h,k} \rangle_{D_T} \end{aligned} \quad (4.25)$$

and

$$\mu_0 \left\langle \frac{\partial \mathbf{H}_{h,k}}{\partial t}, \boldsymbol{\zeta}_h \right\rangle_{\tilde{D}_T} + \sigma \left\langle \nabla \times \widetilde{\mathbf{H}}_{h,k}, \nabla \times \boldsymbol{\zeta}_h \right\rangle_{\tilde{D}_T} = -\mu_0 \langle \mathbf{v}_{h,k}, \boldsymbol{\zeta}_h \rangle_{\tilde{D}_T}. \quad (4.26)$$

In order to prove that  $\mathbf{m}$  and  $\mathbf{H}$  satisfy (2.3) and (2.4), respectively, we prove that as  $h$  and  $k$  tend to 0 there hold

$$\langle \mathbf{m}_{h,k}^- \times \mathbf{v}_{h,k}, \mathbf{w}_{h,k} \rangle_{D_T} \rightarrow \langle \mathbf{m} \times \mathbf{m}_t, \mathbf{m} \times \boldsymbol{\phi} \rangle_{D_T}, \quad (4.27)$$

$$\langle \mathbf{v}_{h,k}, \mathbf{w}_{h,k} \rangle_{D_T} \rightarrow \langle \mathbf{m}_t, \mathbf{m} \times \boldsymbol{\phi} \rangle_{D_T}, \quad (4.28)$$

$$\langle \nabla \mathbf{m}_{h,k}^-, \nabla \mathbf{w}_{h,k} \rangle_{D_T} \rightarrow \langle \nabla \mathbf{m}, \nabla(\mathbf{m} \times \boldsymbol{\phi}) \rangle_{D_T}, \quad (4.29)$$

$$k \langle \nabla \mathbf{v}_{h,k}, \nabla \mathbf{w}_{h,k} \rangle_{D_T} \rightarrow 0, \quad (4.30)$$

$$\langle \widetilde{\mathbf{H}}_{h,k}, \mathbf{w}_{h,k} \rangle_{D_T} \rightarrow \langle \mathbf{H}, \mathbf{m} \times \boldsymbol{\phi} \rangle_{D_T}, \quad (4.31)$$

and

$$\left\langle \frac{\partial \mathbf{H}_{h,k}}{\partial t}, \boldsymbol{\zeta}_h \right\rangle_{\tilde{D}_T} \rightarrow \langle \mathbf{H}_t, \boldsymbol{\zeta} \rangle_{\tilde{D}_T}, \quad (4.32)$$

$$\left\langle \nabla \times \widetilde{\mathbf{H}}_{h,k}, \nabla \times \boldsymbol{\zeta}_h \right\rangle_{\tilde{D}_T} \rightarrow \langle \nabla \times \mathbf{H}, \nabla \times \boldsymbol{\zeta} \rangle_{\tilde{D}_T}, \quad (4.33)$$

$$\langle \mathbf{v}_{h,k}, \boldsymbol{\zeta}_h \rangle_{\tilde{D}_T} \rightarrow \langle \mathbf{m}_t, \boldsymbol{\zeta} \rangle_{\tilde{D}_T}. \quad (4.34)$$

We now prove (4.27) and (4.30); the others can be obtained in the same manner.

Using the triangular inequality and Holder's inequality, we estimate

$$I_{h,k} := \left| \langle \mathbf{m}_{h,k}^- \times \mathbf{v}_{h,k}, \mathbf{w}_{h,k} \rangle_{D_T} - \langle \mathbf{m} \times \mathbf{m}_t, \mathbf{m} \times \boldsymbol{\phi} \rangle_{D_T} \right|$$

as follows:

$$\begin{aligned} I_{h,k} & \leq \left| \langle \mathbf{m}_{h,k}^- \times \mathbf{v}_{h,k}, \mathbf{w}_{h,k} - \mathbf{m}_{h,k}^- \times \boldsymbol{\phi} \rangle_{D_T} \right| + \left| \langle \mathbf{m}_{h,k}^- \times \mathbf{v}_{h,k}, (\mathbf{m}_{h,k}^- - \mathbf{m}) \times \boldsymbol{\phi} \rangle_{D_T} \right| \\ & \quad + \left| \langle (\mathbf{m}_{h,k}^- - \mathbf{m}) \times \mathbf{v}_{h,k}, \mathbf{m} \times \boldsymbol{\phi} \rangle_{D_T} \right| + \left| \langle \mathbf{m} \times (\mathbf{v}_{h,k} - \mathbf{m}_t), \mathbf{m} \times \boldsymbol{\phi} \rangle_{D_T} \right| \\ & \leq \|\mathbf{m}_{h,k}^-\|_{\mathbb{L}^\infty(D_T)} \|\mathbf{v}_{h,k}\|_{D_T} \|\mathbf{w}_{h,k} - \mathbf{m}_{h,k}^- \times \boldsymbol{\phi}\|_{D_T} \\ & \quad + \|\mathbf{m}_{h,k}^-\|_{\mathbb{L}^\infty(D_T)} \|\mathbf{v}_{h,k}\|_{D_T} \|\mathbf{m} - \mathbf{m}_{h,k}^-\|_{D_T} \|\boldsymbol{\phi}\|_{\mathbb{L}^\infty(D_T)} \\ & \quad + \|\mathbf{m} - \mathbf{m}_{h,k}^-\|_{D_T} \|\mathbf{v}_{h,k}\|_{D_T} \|\boldsymbol{\phi}\|_{\mathbb{L}^\infty(D_T)} \\ & \quad + \|\mathbf{v}_{h,k} - \mathbf{m}_t\|_{D_T} \|\boldsymbol{\phi}\|_{\mathbb{L}^\infty(D_T)} \\ & \leq c \left( \|\mathbf{w}_{h,k} - \mathbf{m}_{h,k}^- \times \boldsymbol{\phi}\|_{D_T} + \|\mathbf{m} - \mathbf{m}_{h,k}^-\|_{D_T} + \|\mathbf{v}_{h,k} - \mathbf{m}_t\|_{D_T} \right), \end{aligned}$$

where we have used (4.21) and Lemma 4.1, noting Remark 4.2. The interpolation operators  $I_{\mathbb{V}_h}$  and  $I_{\mathbb{Y}_h}$  have the following properties (see e.g., [5] and [14])

$$\begin{aligned} \|\mathbf{m}_{h,k}^- \times \boldsymbol{\phi} - \mathbf{w}_{h,k}\|_{\mathbb{L}^2([0,T], \mathbb{H}^1(D))} &\leq Ch \|\mathbf{m}_{h,k}^-\|_{\mathbb{H}^1(D_T)} \|\boldsymbol{\phi}\|_{\mathbb{W}^{2,\infty}(D_T)}, \\ \|\boldsymbol{\zeta}(t) - \boldsymbol{\zeta}_h(t)\|_{\tilde{D}} + \|\nabla \times (\boldsymbol{\zeta}(t) - \boldsymbol{\zeta}_h(t))\|_{\tilde{D}} &\leq Ch \|\nabla^2 \boldsymbol{\zeta}\|_{\tilde{D}}. \end{aligned} \quad (4.35)$$

This implies

$$\lim_{k,h \rightarrow 0} I_{h,k} = 0,$$

proving (4.27).

In order to prove (4.30) we first note that

$$\begin{aligned} \|\nabla \mathbf{w}_{h,k}\|_{D_T} &\leq \|\nabla(\mathbf{m}_{h,k}^- \times \boldsymbol{\phi} - \mathbf{w}_{h,k})\|_{D_T} + \|\nabla(\mathbf{m}_{h,k}^- \times \boldsymbol{\phi})\|_{D_T} \\ &\leq ch \|\mathbf{m}_{h,k}^-\|_{\mathbb{H}^1(D_T)} \|\boldsymbol{\phi}\|_{\mathbb{W}^{2,\infty}(D_T)} + \|\nabla \mathbf{m}_{h,k}^-\|_{D_T} \|\nabla \boldsymbol{\phi}\|_{\mathbb{L}^\infty(D_T)} \\ &\leq c \|\boldsymbol{\phi}\|_{\mathbb{W}^{2,\infty}(D_T)}, \end{aligned}$$

where we have used (4.35) and the boundedness of  $\|\mathbf{m}_{h,k}^-\|_{\mathbb{H}^1(D_T)}$ . Now using Holder's inequality we obtain

$$k \langle \nabla \mathbf{v}_{h,k}, \nabla \mathbf{w}_{h,k} \rangle_{D_T} \leq ck \|\nabla \mathbf{v}_{h,k}\|_{D_T}. \quad (4.36)$$

It is straightforward from (4.8) that  $\{\|\nabla \mathbf{v}_{h,k}\|_{D_T}\}$  is bounded when  $\theta \in (\frac{1}{2}, 1]$ . Hence, taking the limit as  $k$  and  $h$  tend to 0 in (4.36) yields (4.30) for these values of  $\theta$ .

When  $\theta \in [0, \frac{1}{2}]$ , using the inverse estimate we obtain

$$\sum_{i=0}^{j-1} \|\nabla \mathbf{v}_h^{(i+1)}\|_D^2 \leq ch^{-2} \sum_{i=0}^{j-1} \|\mathbf{v}_h^{(i+1)}\|_D^2,$$

or equivalently,

$$\|\nabla \mathbf{v}_{h,k}\|_{D_T} \leq ch^{-1} \|\mathbf{v}_{h,k}\|_{D_T}.$$

Hence under the assumption (4.24), the inequality (4.36) becomes

$$k \langle \nabla \mathbf{v}_{h,k}, \nabla \mathbf{w}_{h,k} \rangle_{D_T} \leq ckh^{-1} \|\mathbf{v}_{h,k}\|_{D_T} \leq ckh^{-1}$$

when  $\theta \in [0, 1/2]$ . Therefore, under the assumption (4.24) there holds

$$k \langle \nabla \mathbf{v}_{h,k}, \nabla \mathbf{w}_{h,k} \rangle_{D_T} \rightarrow 0.$$

We now prove (2.5). Since  $\mathbf{m}_h^0 = I_{\mathbb{V}_h}(\mathbf{m}_0)$ , the sequence  $\{\mathbf{m}_h^0\}$  converges to  $\mathbf{m}_0$  in  $\mathbb{L}^2(D)$  as  $h$  tends to 0. Using the weak continuity of the trace operator we obtain that  $\mathbf{m}(0, \cdot) = \mathbf{m}_0$  in the sense of traces.

Finally, applying weak lower semicontinuity of norms in inequality (4.1) we obtain the energy inequality (2.6), which completes the proof.  $\square$



## 5 Numerical experiments

In order to carry out physically relevant experiments, the initial fields  $\mathbf{m}_0$ ,  $\mathbf{H}_0$  must satisfy condition (2.2). This can be achieved by taking

$$\mathbf{H}_0 = \mathbf{H}_0^* - \chi_D \mathbf{m}_0,$$

where  $\operatorname{div} \mathbf{H}_0^* = 0$  in  $\tilde{D}$ . In our experiment, for simplicity, we choose  $\mathbf{H}_0^*$  to be a constant. We solve an academic example with  $D = \tilde{D} = (0, 1)^3$  and

$$\begin{aligned} \mathbf{m}_0(\mathbf{x}) &= \begin{cases} (0, 0, -1), & |\mathbf{x}^*| \geq \frac{1}{2}, \\ (2\mathbf{x}^* A, A^2 - |\mathbf{x}^*|^2)/(A^2 + |\mathbf{x}^*|^2), & |\mathbf{x}^*| \leq \frac{1}{2}, \end{cases} \\ \mathbf{H}_0^*(\mathbf{x}) &= (0, 0, H_s), \quad \mathbf{x} \in \tilde{D}, \end{aligned}$$

where  $\mathbf{x} = (x_1, x_2, x_3)$ ,  $\mathbf{x}^* = (x_1 - 0.5, x_2 - 0.5, 0)$  and  $A = (1 - 2|\mathbf{x}^*|)^4/4$ . The constant  $H_s$  represents the strength of  $\mathbf{H}_0$  in the  $x_3$ -direction. We compute the experiments for  $H_s = 0, \pm 30, \pm 100$  and  $\pm 1000$ . We set the values for the other parameters in (1.1) and (1.2) as  $\lambda_1 = \lambda_2 = \mu_0 = \sigma = 1$ .

The domain  $D$  is partitioned into uniform cubes with the mesh size  $h = 1/2^3$ , where each cube consists of six tetrahedra. We choose the time step  $k = 10^{-3}$  and the parameter  $\theta$  in Algorithm 2.1 to be 0.7. The construction of the basis functions for  $\mathbb{W}_h^{(j)}$  and  $\mathbb{Y}_h$  in this algorithm is discussed in [13]. At each iteration we need to solve a linear system of size  $(2N + M) \times (2N + M)$ , recalling that  $N$  is the number of vertices and  $M$  is the number of edges in the triangulation. The code is written in Fortran90.

The evolution of  $\|\nabla \mathbf{m}_{h,k}\|_D$ ,  $\|\mathbf{H}_{h,k}\|_{\tilde{D}}$  and  $\|\nabla \times \mathbf{H}_{h,k}\|_{\tilde{D}}$  are depicted in Figures 1, 2 and 3, respectively. Figure 4 shows that the solution satisfies condition (2.6) in Definition 2.1.

**Remark 5.1.** *By the time this paper was written up, we learnt that Bañas, Page and Praetorius [4] independently solved a similar problem. They also used a linear scheme similar to our scheme, even though their variational formulation was different.*

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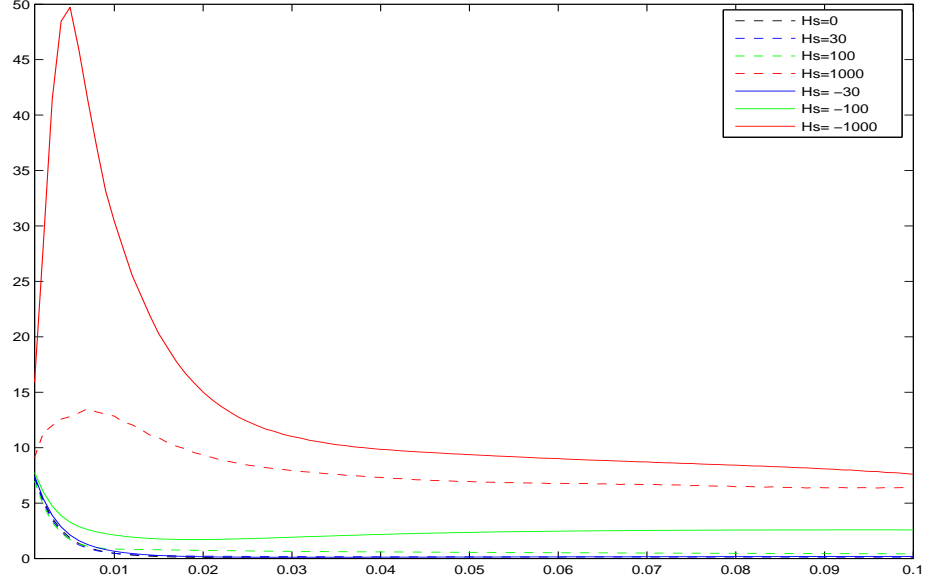


Figure 1: Plot of  $t \mapsto \|\nabla \mathbf{m}_{h,k}(t)\|_D$ .

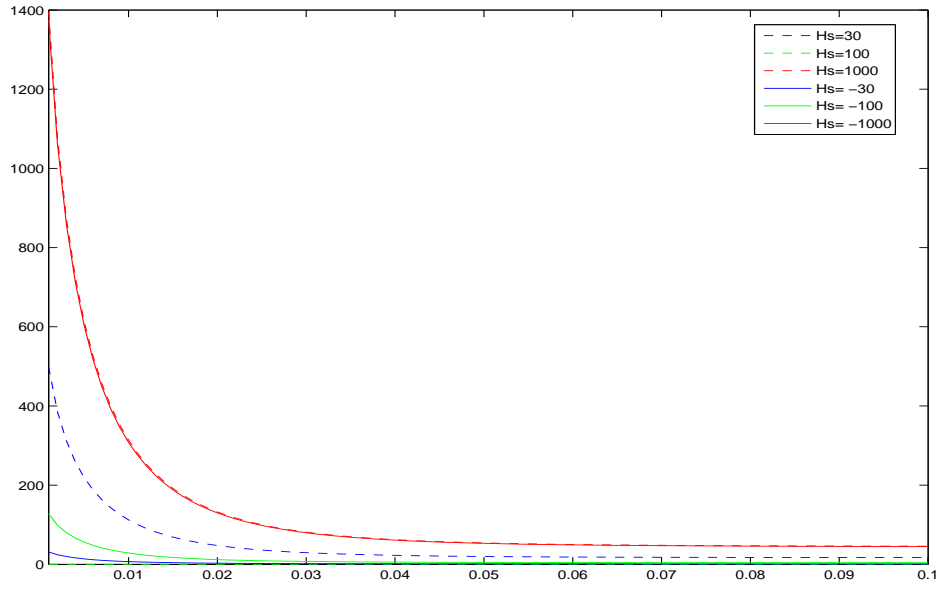


Figure 2: Plot of  $t \mapsto \|\mathbf{H}_{h,k}(t)\|_{\tilde{D}}$ .

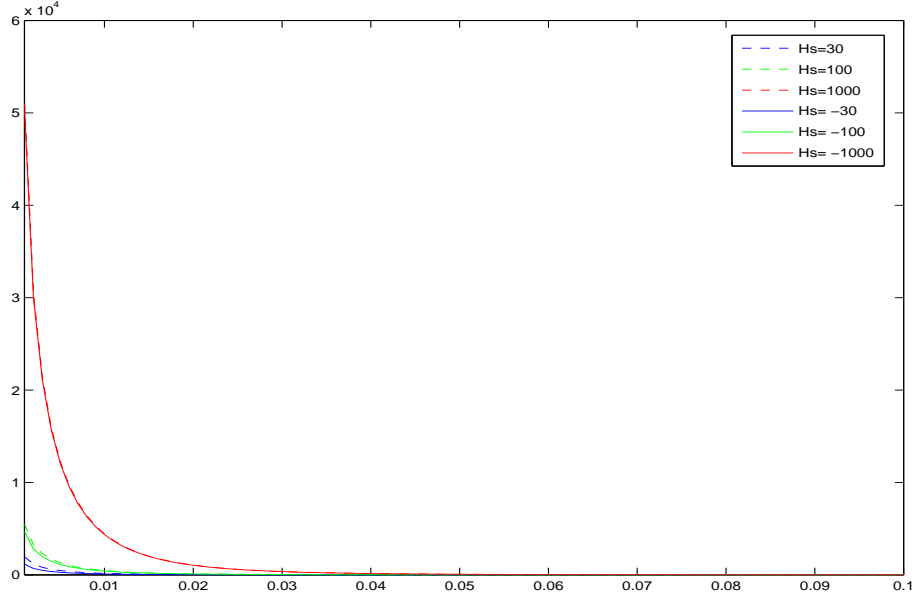


Figure 3: Plot of  $t \mapsto \|\nabla \times \mathbf{H}_{h,k}(t)\|_{\tilde{D}}$ .

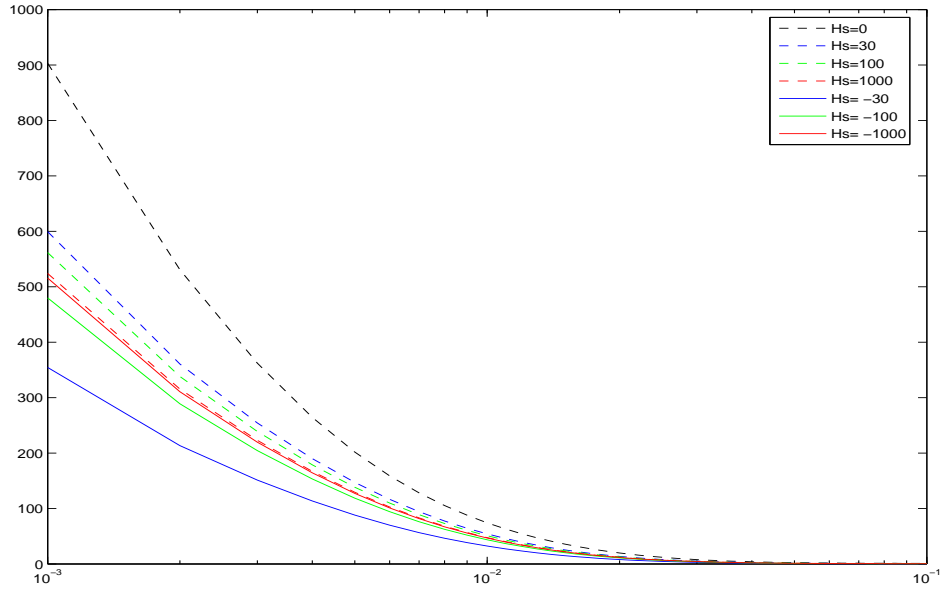


Figure 4: Plot of  $\log t \mapsto \mathcal{E}(t)$

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